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About a system of anti-periodic trigonometric functions

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ABSTRACT

In this paper we introduce the so-called second kind trigonometric system, which is a useful tool for the representation of 2π "anti-periodic" functions. Using these functions we study a sequence of trigonometric polynomials, which are bi-orthogonal in the Szegő's sense. We study the usual topics in the theory of Fourier series and we present a new connection with the orthogonal polynomials (OP) on the unit circle and some useful properties like: recurrence relations, kernel representations and a Favard's type theorem.

1. Introduction

The relation between sequences of orthogonal polynomials (OP) on the interval $[-1, 1]$ and on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is based on the Joukowski transformation. In particular, the sequence of Szegő polynomials $\{z^n\}$, which is orthogonal with respect to the Lebesgue measure on $\mathbb{T}$ (see [1]), is related in the following way, for $x = \cos\theta$ and $z = e^{i\theta}$, with the sequences of polynomials:

$$F_{n,a}(x) = \frac{z^{n+a} + z^{-(n+a)}}{z^a + z^{-a}} \quad \text{and} \quad G_{n,b}(x) = \frac{z^{n+b} - z^{-(n+b)}}{z^b - z^{-b}},$$

which are orthogonal on $[-1, 1]$ for $a = 0, \frac{1}{2}$ and $b = \frac{1}{2}, 1$, (see [2]).

Indeed,

(i) for $a = 0$, $F_{n,0}(x) = T_n(x)$ (Chebyshev polynomials of first kind),
(ii) for $a = \frac{1}{2}$, $F_{n,\frac{1}{2}}(x) = W_n(x)$ (Chebyshev polynomials of third kind),
(iii) for $b = 1$, $G_{n,1}(x) = U_n(x)$ (Chebyshev polynomials of second kind),
(iv) for $b = \frac{1}{2}$, $G_{n,\frac{1}{2}}(x) = V_n(x)$ (Chebyshev polynomials of fourth kind).

Therefore the following trigonometric expressions $\cos n\theta$, $\frac{\sin(n+1)\theta}{\sin\theta}$, $\frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{\theta}{2}}$ and $\frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}$ play an important role in the above relations. We recall that the weight of orthogonality for the Chebyshev polynomials of third and fourth kind are $\frac{1}{\pi}\sqrt{\frac{1+x}{1-x}}$ and $\frac{1}{\pi}\sqrt{\frac{1-x}{+x}}$ respectively.

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On the other hand, it is very well-known that the trigonometric system $\{\cos n\theta\}_{n=0}^{\infty} \bigcup \{\sin n\theta\}_{n=1}^{\infty}$ is a maximal orthogonal set in the space $L^2([-\pi.\pi])$ with respect to the Lebesgue measure and it can also be considered a bi-orthogonal system in the Szegő' sense, (see [3]). In this work we consider another bi-orthogonal system, which is related with the above expressions. Indeed we consider the system $\{\cos(n+\frac{1}{2})\theta\}_{n=0}^{\infty} \bigcup \{\sin(n+\frac{1}{2})\theta\}_{n=0}^{\infty}$, that we call the second kind trigonometric system. Our aim is to establish the properties of this new system and to prove that it is useful for the representation of 2π anti-periodic functions. This system is also used to generate a bi-orthogonal system of trigonometric functions which can be connected with a family of OP on the unit circle. The usual properties like recurrence relations, kernel representations and a Favard's type theorem are also studied.

The organization of the paper is the following. In Section 2 we present a generalization of the classical trigonometric system that we call the second kind trigonometric system, which is a maximal orthogonal set in $L^2([-\pi, \pi])$. We prove that the set of its linear combinations is dense in the space of the real 2π anti-periodic functions. In Section 3 we present a bi-orthogonal system of trigonometric functions in the Szegő's sense. We obtain the five term recurrence relation satisfied by this system as well as the coefficients of this relation in terms of the associated monic orthogonal polynomial sequence on the unit circle. We also obtain a Christoffel–Darboux type formula and we connect the kernels for both systems. Finally, Section 4 is devoted to obtain another recurrence relation which is needed to prove a Favard's type theorem. This last theorem is the main result of this section.

2. Second kind trigonometric system of anti-periodic trigonometric polynomials

A natural generalization of the classical trigonometric system ordered as follows

$$\{1, \sin\theta, \cos\theta, \ldots, \sin n\theta, \cos n\theta, \ldots\}$$

is the following system of functions

$$\{\sin a\theta, \cos a\theta, \sin(1+a)\theta, \cos(1+a)\theta, \ldots, \sin(n+a)\theta, \cos(n+a)\theta, \ldots\}.$$

Since it is known that the trigonometric system is a maximal orthogonal set in the space $L^2([-\pi, \pi])$ with the Lebesgue measure, now we want to obtain the values of a for which this new system is also a maximal orthogonal set in the same space.

Theorem 1. *The system*

$$\{\sin a\theta, \cos a\theta, \sin(1+a)\theta, \cos(1+a)\theta, \ldots, \sin(n+a)\theta, \cos(n+a)\theta, \ldots\}$$

with $a \neq 0$ is orthogonal in the space $L^2([-\pi, \pi])$ if and only if $a = \frac{m}{2}$ with $m \in \mathbb{N}$.

Moreover, it is a maximal orthogonal set if and only if $a = \frac{1}{2}$.

Proof. The proof of the orthogonality is straightforward.

For proving the maximal character, take $a = \frac{1}{2}$ and let f be a function in $L^2([-\pi, \pi])$ and such that

$$\int_{-\pi}^{\pi} f(\theta) \sin\left(n+\frac{1}{2}\right)\theta \mathrm{d}\theta = \int_{-\pi}^{\pi} f(\theta) \cos\left(n+\frac{1}{2}\right)\theta \mathrm{d}\theta = 0$$

for $n \geq 0$. Taking into account the well-known trigonometric formulas $\sin(n+\frac{1}{2})\theta + \sin(n-\frac{1}{2})\theta = 2\sin n\theta \cos\frac{\theta}{2}$ and $\cos(n+\frac{1}{2})\theta + \cos(n-\frac{1}{2})\theta = 2\cos n\theta \cos\frac{\theta}{2}$, it follows that

$$\int_{-\pi}^{\pi} f(\theta) \cos\frac{\theta}{2} \sin n\theta \mathrm{d}\theta = 0 \quad \text{for } n \geq 1,$$

$$\int_{-\pi}^{\pi} f(\theta) \cos\frac{\theta}{2} \cos n\theta \mathrm{d}\theta = 0 \quad \text{for } n \geq 0.$$

From the completeness of the classical trigonometric system we get that $f(\theta)\cos\frac{\theta}{2} = 0$, *a.e.* in $[-\pi, \pi]$, from which it follows that $f = 0$ *a.e.* in $[-\pi, \pi]$. Besides, for $a = \frac{m}{2}$ with $m \geq 2$, it is clear that the corresponding set is not maximal because we can add, at least, $\cos\frac{\theta}{2}$ and $\sin\frac{\theta}{2}$ and the new set is also orthogonal. $\square$

Remark 1. To prove the maximal orthogonal character of the system $\{\sin(n+\frac{1}{2})\theta, \cos(n+\frac{1}{2})\theta\}_{n=0}^{\infty}$, which we will call second kind trigonometric system, we can take into account, in an alternative way, that they are the eigenfunctions of the following regular Sturm–Liouville problem:

$$x''(\theta) + \lambda x(\theta) = 0, \quad \text{with } x(\pi) + x(-\pi) = 0 \quad \text{and} \quad x'(\pi) + x'(-\pi) = 0.$$

In the sequel our aim is to prove that the set of the linear combinations of functions of the second kind trigonometric system is dense in the space of the real 2π anti-periodic functions, that we define as follows.

Definition 1. We say that a real function f is 2π anti-periodic if $f(x+2\pi) = -f(x),\ \forall x$.

For obtaining this result we follow the steps of the proof given in [4] concerning the density of the trigonometric system.

Let f be a real continuous and 2π anti-periodic function. Since the restriction of the function f to the interval $[-\pi, \pi]$ belongs to $L^2([-\pi, \pi])$ and taking into account the maximality of the second kind trigonometric system $\{\cos(n+\frac{1}{2})\theta\}_{n=0}^{\infty} \bigcup \{\sin(n+\frac{1}{2})\theta\}_{n=0}^{\infty}$ we get that f can be expanded in a Fourier series which converges to f in the L^2-norm. Therefore we can consider the nth partial sum of the Fourier series of f defined by

$$\begin{aligned} S_n(x,f) &= \sum_{k=0}^{n} \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\left(\sin\left(k+\frac{1}{2}\right)x \sin\left(k+\frac{1}{2}\right)s + \cos\left(k+\frac{1}{2}\right)x\cos\left(k+\frac{1}{2}\right)s\right)\mathrm{d}s \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\sum_{k=0}^{n}\cos\left(\left(k+\frac{1}{2}\right)(x-s)\right)\mathrm{d}s = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)E_n(x-s)\mathrm{d}s, \end{aligned} \tag{1}$$

where $E_n(t) = \sum_{k=0}^{n}\cos(k+\frac{1}{2})t$ is a Dirichlet type kernel which can be represented as follows

$$E_n(t) = \begin{cases} \dfrac{\sin(n+1)t}{2\sin(t/2)}, & 0 < |t| < \pi, \\ n+1, & t = 0. \end{cases}$$

It is immediate to see that $E_n(t+2\pi) = -E_n(t)$. Moreover, if in (1) we do the change $z = -x + s$ and take into account that E_n and f are 2π anti-periodic functions we can rewrite (1) as a convolution integral

$$S_n(x,f) = \frac{1}{\pi}\int_{-x-\pi}^{-x+\pi} f(x+z)E_n(z)\mathrm{d}z = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+z)E_n(z)\mathrm{d}z = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-z)E_n(z)\mathrm{d}z.$$

From the expression of $E_n(t)$ we get

$$\frac{1}{\pi}\int_{-\pi}^{\pi} E_n(z)\mathrm{d}z = \frac{4}{\pi}\sum_{k=0}^{n}(-1)^k\frac{1}{2k+1},$$

that is, the nth partial sum of an alternate convergent series that we are going to denote by e_n, that is,

$$e_n = \frac{4}{\pi}\sum_{k=0}^{n}(-1)^k\frac{1}{2k+1}. \tag{2}$$

Another sum which plays an important role in this proof is the following Cesáro type mean. Let us denote by $C_n(x,f) = \frac{1}{n}\sum_{k=0}^{n-1} S_k(x,f)$, that is,

$$C_n(x,f) = \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+z)E_k(z)\mathrm{d}z = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+z)\frac{1}{n}\left(\sum_{k=0}^{n-1}E_k(z)\right)\mathrm{d}z = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+z)L_n(z)\mathrm{d}z,$$

where we denote

$$L_n(t) = \frac{1}{n}\sum_{k=0}^{n-1}E_k(t). \tag{3}$$

It is easy to see that it can be represented by

$$L_n(t) = \begin{cases} \dfrac{\sin(\frac{n+1}{2})t\sin(n\frac{t}{2})}{2n\sin^2(t/2)}, & 0 < |t| < \pi, \\ \dfrac{n+1}{2}, & t = 0. \end{cases}$$

Some immediate properties about these new kernels are gathered in the following Lemmas, which are needed for the proof of the main theorem of this section, Theorem 2.

Lemma 1. *If we denote by* $d_n = \frac{1}{\pi}\int_{-\pi}^{\pi} L_n(z)\mathrm{d}z$*, then*

(i) $d_n = \frac{1}{n}\sum_{k=0}^{n-1} e_k$.
(ii) $d_n \geq \frac{8}{9\pi}$.

Proof. (i) It is immediate from (2) and (3).
(ii) From (i) and using that $e_n \geq \frac{8}{9\pi}$ we get (ii). □

Lemma 2. *If* $E_n(t)$ *and* $L_n(t)$ *are defined as above, then*

(i) $\frac{1}{\pi}\int_{-\pi}^{\pi} |E_n(t)|\mathrm{d}t \leq \frac{4(n+1)}{\pi}$.
(ii) $\frac{1}{\pi}\int_{-\pi}^{\pi} |L_n(t)|\mathrm{d}t \leq \frac{5}{3}$.

Proof. (i) From the definition of $E_n(t)$ we get

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|E_n(t)|\mathrm{d}t \le \frac{1}{\pi}\sum_{k=0}^{n}\int_{-\pi}^{\pi}\left|\cos\left(k+\frac{1}{2}\right)t\right|\mathrm{d}t.$$

Now we compute the value of $\int_{-\pi}^{\pi}\left|\cos\left(k+\frac{1}{2}\right)t\right|\mathrm{d}t$ by taking into account the even character of the integrand and the zeros $t_r = \frac{(2r+1)\pi}{2k+1}$ for $r = 0,\dots,k$ of the integrand in the interval $[0,\pi]$, that is,

$$\begin{aligned}\int_{-\pi}^{\pi}\left|\cos\left(k+\frac{1}{2}\right)t\right|\mathrm{d}t &= 2\int_{0}^{\pi}\left|\cos\left(k+\frac{1}{2}\right)t\right|\mathrm{d}t = 2\left(\int_{0}^{\frac{\pi}{2k+1}}\left|\cos\left(k+\frac{1}{2}\right)t\right|\mathrm{d}t\right.\\ &\quad\left.+\sum_{r=1}^{k}\int_{\frac{(2r-1)\pi}{2k+1}}^{\frac{(2r+1)\pi}{2k+1}}\left|\cos\left(k+\frac{1}{2}\right)t\right|\mathrm{d}t\right)\\ &= 2\left(\frac{2}{2k+1}+\frac{4k}{2k+1}\right) = 4.\end{aligned}$$

Therefore

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|E_n(t)|\mathrm{d}t \le \frac{4(n+1)}{\pi},$$

which proves (i).

(ii) From (i) it follows that

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left|\frac{\sin(nt)}{2n\sin(t/2)}\right|\mathrm{d}t \le \frac{4}{\pi}.$$

Using the representation of $L_n(t)$ for $t \neq 0$ and a well-known trigonometric expression we get

$$L_n(t) = \frac{\sin(\frac{n+1}{2})t\sin(n/2)t}{2n\sin^2(t/2)} = \frac{\sin^2(nt/2)}{2n\sin^2(t/2)}\cos\left(\frac{t}{2}\right)+\frac{1}{2}\frac{\sin(nt)}{2n\sin(t/2)} = K_n(t)\cos\left(\frac{t}{2}\right)+\frac{1}{2}\frac{\sin(nt)}{2n\sin(t/2)},$$

where $K_n(t)$ is the Féjer kernel, (see [5]).

Hence $|L_n(t)| \le K_n(t)+\frac{1}{2}|\frac{\sin(nt)}{2n\sin(t/2)}|$ and taking into account that $\frac{1}{\pi}\int_{-\pi}^{\pi}K_n(t)\mathrm{d}t = 1$, (see [5]), we obtain $\frac{1}{\pi}\int_{-\pi}^{\pi}|L_n(t)|\mathrm{d}t \le 1+\frac{2}{\pi} < \frac{5}{3}$. □

Theorem 2. *Let f be a real continuous 2π anti-periodic function and let $C_n(x,f)$ be the nth Cesáro type mean defined above. Then the sequence $\{\frac{1}{d_n}C_n(x,f)\}$ converges to f uniformly in the real line.*

Proof. Let $\varepsilon > 0$ and let $x \in [-\pi,\pi]$, then

$$\begin{aligned}\left|\frac{1}{d_n}C_n(x,f)-f(x)\right| &= \left|\frac{1}{d_n}C_n(x,f)-\frac{f(x)}{\pi d_n}\int_{-\pi}^{\pi}L_n(z)\mathrm{d}z\right| = \left|\frac{1}{\pi d_n}\int_{-\pi}^{\pi}(f(x+z)-f(x))L_n(z)\mathrm{d}z\right|\\ &\le \frac{1}{\pi d_n}\int_{-\pi}^{\pi}|f(x+z)-f(x)||L_n(z)|\mathrm{d}z \le \frac{9}{8}\int_{-\pi}^{\pi}|f(x+z)-f(x)||L_n(z)|\mathrm{d}z,\end{aligned}$$

where the last inequality follows from Lemma 1.

Applying the uniform continuity of f we obtain that there exists δ such that $|f(x+z)-f(x)| < \frac{\varepsilon}{15}$ for $|z| < \delta$. Besides, since f is bounded on $[-\pi,\pi]$ we get that there exists M such that $|f(x+z)-f(x)| \le 2\max_{-\pi\le z\le\pi}|f(z)| = M$.

Therefore

$$\begin{aligned}\left|\frac{1}{d_n}C_n(x,f)-f(x)\right| &\le \frac{9\pi}{8}\left(\frac{1}{\pi}\int_{-\delta}^{\delta}|f(x+z)-f(x)||L_n(z)|\mathrm{d}z+\frac{1}{\pi}\int_{[-\pi,-\delta]\cup[\delta,\pi]}|f(x+z)-f(x)||L_n(z)|\mathrm{d}z\right)\\ &\le \frac{9\pi}{8}\left(\frac{\varepsilon}{15\pi}\int_{-\delta}^{\delta}|L_n(z)|\mathrm{d}z+\frac{M}{\pi}\int_{[-\pi,-\delta]\cup[\delta,\pi]}|L_n(z)|\mathrm{d}z\right) \le \frac{9\pi}{8}\left(\frac{\varepsilon}{9}+\frac{2M}{\pi}\int_{\delta}^{\pi}|L_n(z)|\mathrm{d}z\right),\end{aligned}$$

in accordance with Lemma 2.

Now, taking into account that $\sin\left(\frac{z}{2}\right) > \frac{z}{\pi}$ for $z \in [0,\pi]$ and $\sin^2\left(\frac{z}{2}\right) > \left(\frac{z}{\pi}\right)^2 > \left(\frac{\delta}{\pi}\right)^2$ for $z \in [\delta,\pi]$, we get

$$\int_{\delta}^{\pi}|L_n(z)|\mathrm{d}z = \int_{\delta}^{\pi}\left|\frac{\sin(\frac{n+1}{2})z\sin(n\frac{z}{2})}{2n\sin^2(z/2)}\right|\mathrm{d}z \le \frac{1}{2n}\int_{\delta}^{\pi}\frac{\mathrm{d}z}{(\delta/\pi)^2} = \frac{1}{2n}\left(\frac{\pi}{\delta}\right)^2(\pi-\delta).$$

Finally

$$\left|\frac{1}{d_n}C_n(x,f)-f(x)\right| \le \frac{9\pi}{8}\left(\frac{\varepsilon}{9}+\frac{2M}{\pi}\left(\frac{\pi}{\delta}\right)^2\frac{(\pi-\delta)}{2n}\right) < \varepsilon,$$

for $n \ge n_0$, where $n_0 = E[\frac{9M(\pi-\delta)\pi^2}{4\varepsilon\delta^2}]$. □

3. A bi-orthogonal system of trigonometric functions

Let μ be a finite positive Borel measure on $[-\pi,\pi]$ with infinite support. We orthogonalize the system

$$\left\{\sin\frac{\theta}{2},\cos\frac{\theta}{2},\sin\left(1+\frac{1}{2}\right)\theta,\cos\left(1+\frac{1}{2}\right)\theta,\ldots,\sin\left(n+\frac{1}{2}\right)\theta,\cos\left(n+\frac{1}{2}\right)\theta,\ldots\right\}$$

applying the Gram–Schmidt method. We obtain a new system of trigonometric functions and we choose 2^{-n} as leading coefficient of these new functions in $\sin(n+\frac{1}{2})\theta$ and $\cos(n+\frac{1}{2})\theta$, so we get that they are given by:

$$F_{2n}\left(\frac{\theta}{2}\right)=\frac{1}{2^n}\sin\left(n+\frac{1}{2}\right)\theta+ldt,\quad n=0,1,\ldots$$
$$F_{2n-1}\left(\frac{\theta}{2}\right)=\frac{1}{2^{n-1}}\cos\left(n-1+\frac{1}{2}\right)\theta+ldt,\quad n=1,2,\ldots,$$

where we denote by ldt the lower degree terms in the sense of the order fixed above. If we interpret the measure μ as a measure supported on the unit circle $\mathbb{T}$, we can consider the monic OP sequence related to μ (MOPS(μ)), $\{\Phi_n(z)\}$, and we can connect the bi-orthogonal system of trigonometric functions $\{F_n(\frac{\theta}{2})\}$ with the MOPS(μ).

We recall that $\{\Phi_n(z)\}$ is characterized by the following conditions:

(1) $\Phi_n(z)$ is a polynomial of degree n and its leading coefficient is 1.
(2) $\langle\Phi_n(z),\Phi_m(z)\rangle_\mu=\int_{\mathbb{T}}\Phi_n(z)\overline{\Phi_m(z)}\mathrm{d}\mu(z)=K_n\delta_{n,m}$, with $K_n\neq 0,\ \forall n$.

We also recall (see [3]) that a system of trigonometric polynomials $\{A_n(\theta),B_n(\theta)\}$ is called bi-orthogonal with respect to a weight function $\omega(\theta)$ if the following conditions hold:

(1) $A_n(\theta)$ and $B_n(\theta)$ are linearly independent polynomials of degree n.
(2) They satisfy the orthogonality conditions:

$$\int_{-\pi}^{\pi}A_n(\theta)A_m(\theta)\omega(\theta)\mathrm{d}\theta=\int_{-\pi}^{\pi}B_n(\theta)B_m(\theta)\omega(\theta)\mathrm{d}\theta=K_n\delta_{n,m},\quad\text{with }K_n\neq 0,$$
$$\int_{-\pi}^{\pi}A_n(\theta)B_m(\theta)\omega(\theta)\mathrm{d}\theta=0,\quad n,m=0,1,\ldots.$$

Theorem 3. *Let μ be a finite positive Borel measure on $[-\pi,\pi]$ and let $\{F_n(\frac{\theta}{2})\}$ be the sequence of bi-orthogonal trigonometric functions. Let us consider the measure μ as a measure on $\mathbb{T}$ and let us consider $\{\Phi_n(z)\}$ the MOPS(μ). Then we have for $z=\mathrm{e}^{\mathrm{i}\theta}$*

(i)

$$F_{2n}\left(\frac{\theta}{2}\right)=\frac{1}{i2^{n+1}}\left(z^{-(n-\frac{1}{2})}\Phi_{2n}(z)-z^{n-\frac{1}{2}}\overline{\Phi_{2n}}(z^{-1})\right),\quad n\geq 0,\tag{4}$$

and

$$\|F_{2n}\|_\mu^2=\frac{\|\Phi_{2n}\|_\mu^2(1+\Re\Phi_{2n+1}(0))}{2^{2n+1}},\tag{5}$$

(ii)

$$F_{2n-1}\left(\frac{\theta}{2}\right)=\frac{1}{2^n(1+\Re\Phi_{2n-1}(0))}\left(z^{-(n-\frac{1}{2})}\Phi_{2n-1}(z)+z^{n-\frac{1}{2}}\overline{\Phi_{2n-1}}(z^{-1})\right),\quad n\geq 1.\tag{6}$$

and

$$\|F_{2n-1}\|_\mu^2=\frac{\|\Phi_{2n-1}\|_\mu^2}{2^{2n-1}(1+\Re\Phi_{2n-1}(0))}.\tag{7}$$

Proof. (i) Let us write $\Phi_{2n}(z)=z^{2n}+\sum_{k=0}^{2n-1}a_{2n,k}z^k$. Then

$$\frac{1}{i2^{n+1}}\left(z^{-(n-\frac{1}{2})}\Phi_{2n}(z)-z^{n-\frac{1}{2}}\overline{\Phi_{2n}}(z^{-1})\right)=\frac{1}{2^n}\sin\left(n+\frac{1}{2}\right)\theta+\frac{\Im(a_{2n,2n-1}+a_{2n,0})}{2^n}\cos\left(n-\frac{1}{2}\right)\theta$$
$$+\frac{\Re(a_{2n,2n-1}-a_{2n,0})}{2^n}\sin\left(n-\frac{1}{2}\right)\theta\cdots+\frac{\Im(a_{2n,n}+a_{2n,n-1})}{2^n}\cos\frac{\theta}{2}+\frac{\Re(a_{2n,n}-a_{2n,n-1})}{2^n}\sin\frac{\theta}{2}.$$

On the other hand, from the orthogonality of the sequence $\{\Phi_n(z)\}$ we get

$$\frac{1}{i2^{n+1}}\langle z^{-(n-\frac{1}{2})}\Phi_{2n}(z)-z^{n-\frac{1}{2}}\overline{\Phi_{2n}}(z^{-1}),z^{k+\frac{1}{2}}\pm z^{-(k+\frac{1}{2})}\rangle_\mu=0\quad\text{for }k=0,\ldots,n-1,$$
$$\frac{1}{i2^{n+1}}\langle z^{-(n-\frac{1}{2})}\Phi_{2n}(z)-z^{n-\frac{1}{2}}\overline{\Phi_{2n}}(z^{-1}),z^{n+\frac{1}{2}}-z^{-(n+\frac{1}{2})}\rangle_\mu=\frac{(1+\Re\Phi_{2n+1}(0))\|\Phi_{2n}\|_\mu^2}{i2^n},$$

and the norm is given by (5).

(ii) One can proceed in the same way.

Analogous relations to (4) and (6) appear in [6–8]. □

Now it is clear that the orthonormal functions $f_n\left(\frac{\theta}{2}\right) = \frac{F_n(\frac{\theta}{2})}{\|F_n(\frac{\theta}{2})\|_\mu}$ can be computed by

$$f_{2n}\left(\frac{\theta}{2}\right) = \frac{z^{-(n-\frac{1}{2})}\Phi_{2n}(z) - z^{n-\frac{1}{2}}\overline{\Phi_{2n}}(z^{-1})}{\sqrt{2(1+\Re\Phi_{2n+1}(0))}i\|\Phi_{2n}\|_\mu}, \tag{8}$$

$$f_{2n-1}\left(\frac{\theta}{2}\right) = \frac{z^{-(n-\frac{1}{2})}\Phi_{2n-1}(z) + z^{n-\frac{1}{2}}\overline{\Phi_{2n-1}}(z^{-1})}{\sqrt{2(1+\Re\Phi_{2n-1}(0))}\|\Phi_{2n-1}\|_\mu}. \tag{9}$$

Theorem 4. *Let $\{f_n(\frac{\theta}{2})\}$ be the sequence of bi-orthonormal trigonometric functions. Then there exist three sequences of coefficients $\{a_n\}_{n\geq 2}$, $\{c_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 0}$ such that for $n \geq 0$ the following five term recurrence relation holds*

(i)

$$\cos\theta f_n\left(\frac{\theta}{2}\right) = a_{n+2}f_{n+2}\left(\frac{\theta}{2}\right) + c_{n+1}f_{n+1}\left(\frac{\theta}{2}\right) + b_nf_n\left(\frac{\theta}{2}\right) + c_nf_{n-1}\left(\frac{\theta}{2}\right) + a_nf_{n-2}\left(\frac{\theta}{2}\right), \tag{10}$$

with the initial conditions $f_{-2}(\frac{\theta}{2}) = 0$, $f_{-1}(\frac{\theta}{2}) = 0$, $f_0(\frac{\theta}{2}) = \frac{1}{\sqrt{m_0}}\sin\frac{\theta}{2}$, and $f_1(\frac{\theta}{2}) = \sqrt{\frac{m_0}{m_0m_2-m_1^2}}(\cos\frac{\theta}{2} - \frac{m_1}{m_0}\sin\frac{\theta}{2})$, where $m_0 = \langle \sin\frac{\theta}{2}, \sin\frac{\theta}{2}\rangle_\mu$, $m_1 = \langle \sin\frac{\theta}{2}, \cos\frac{\theta}{2}\rangle_\mu$ and $m_2 = \langle \cos\frac{\theta}{2}, \cos\frac{\theta}{2}\rangle_\mu$.

(ii) *Moreover, it holds that*

$$a_{n+2}^2 + c_{n+1}^2 + b_n^2 + c_n^2 + a_n^2 < 1, \quad \forall n.$$

Proof. (i) It is clear that we can write

$$\cos\theta f_n\left(\frac{\theta}{2}\right) = \sum_{k=0}^{n+2} a_{k,n}f_k\left(\frac{\theta}{2}\right) \quad \text{and} \quad \cos\theta f_i\left(\frac{\theta}{2}\right) = \sum_{k=0}^{i+2} a_{k,i}f_k\left(\frac{\theta}{2}\right)$$

and the coefficients are given by

$$a_{i,n} = \left\langle \cos\theta f_n\left(\frac{\theta}{2}\right), f_i\left(\frac{\theta}{2}\right)\right\rangle_\mu = \left\langle \cos\theta f_i\left(\frac{\theta}{2}\right), f_n\left(\frac{\theta}{2}\right)\right\rangle_\mu = a_{n,i}.$$

Since $a_{n,i} = 0$ for $n > i+2$, then $a_{i,n} = 0$ for $i < n-2$ and therefore

$$\cos\theta f_n\left(\frac{\theta}{2}\right) = a_{n+2,n}f_{n+2}\left(\frac{\theta}{2}\right) + a_{n+1,n}f_{n+1}\left(\frac{\theta}{2}\right) + a_{n,n}f_n\left(\frac{\theta}{2}\right) + a_{n-1,n}f_{n-1}\left(\frac{\theta}{2}\right) + a_{n-2,n}f_{n-2}\left(\frac{\theta}{2}\right).$$

Now we relate the coefficients. We have $a_{n+2,n} = a_{n,n+2}$ and $a_{n+1,n} = a_{n,n+1}$.

Therefore, in order to simplify the notation we write $a_{n+2,n} = a_{n+2}$, $a_{n+1,n} = c_{n+1}$ and $a_{n,n} = b_n$, from which we get (10).

(ii) Applying (10) and the orthogonality properties we get

$$a_{n+2}^2 + c_{n+1}^2 + b_n^2 + c_n^2 + a_n^2 = \left|\int_0^{2\pi} \cos^2\theta f_n^2\left(\frac{\theta}{2}\right) \mathrm{d}\mu(\theta)\right|.$$

On the other hand we have

$$\left|\int_0^{2\pi} \cos^2\theta f_n^2\left(\frac{\theta}{2}\right) \mathrm{d}\mu(\theta)\right| < \int_0^{2\pi} \left|f_n^2\left(\frac{\theta}{2}\right)\right| \mathrm{d}\mu(\theta) = 1,$$

from which it follows (ii). □

Remark 2. For each n the above recurrence relation can be written as follows:

$$\cos\theta\begin{pmatrix} f_0\left(\frac{\theta}{2}\right) \\ \vdots \\ \vdots \\ f_n\left(\frac{\theta}{2}\right)\end{pmatrix} = \begin{pmatrix} b_0 & c_1 & a_2 & 0 & \cdots & \cdots & \cdots & \cdots \\ c_1 & b_1 & c_2 & a_3 & 0 & \cdots & \cdots & \cdots \\ a_2 & c_2 & b_2 & c_3 & a_4 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & a_{n-1} & c_{n-1} & b_{n-1} & c_n \\ \cdots & \cdots & \cdots & \cdots & 0 & a_n & c_n & b_n \end{pmatrix}\begin{pmatrix} f_0\left(\frac{\theta}{2}\right) \\ \vdots \\ \vdots \\ f_n\left(\frac{\theta}{2}\right)\end{pmatrix}$$

$$+ f_{n+1}\left(\frac{\theta}{2}\right)\begin{pmatrix} 0 \\ \vdots \\ \vdots \\ a_{n+1} \\ c_{n+1}\end{pmatrix} + f_{n+2}\left(\frac{\theta}{2}\right)\begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ a_{n+2}\end{pmatrix}.$$

Notice that the matrix of coefficients is a five diagonal symmetric matrix. This matrix representation together with the recurrence relation given in Theorem 9 suggest a connection with the CMV representation for the multiplication operator by z, (see [9]).

Like in the classical Szegő connection we can relate the Verblunsky coefficients, $\{\Phi_n(0)\}$, and the parameters of the five term recurrence relation. Next we give the corresponding formulas taking into account that the subindices are even or odd numbers.

Theorem 5. *Let $\{f_n(\frac{\theta}{2})\}$ be the orthonormal sequence satisfying relation (10) and let $\{\Phi_n(z)\}$ be the MOPS(μ). Then the following relations hold*

$$2a_{2n+2} = \sqrt{(1-|\Phi_{2n+2}(0)|^2)(1-|\Phi_{2n+1}(0)|^2)}\sqrt{\frac{1+\Re\Phi_{2n+3}(0)}{1+\Re\Phi_{2n+1}(0)}}, \quad n \ge 0, \tag{11}$$

$$2a_{2n+1} = \sqrt{(1-|\Phi_{2n+1}(0)|^2)(1-|\Phi_{2n}(0)|^2)}\sqrt{\frac{1+\Re\Phi_{2n-1}(0)}{1+\Re\Phi_{2n+1}(0)}}, \quad n \ge 1, \tag{12}$$

$$2b_{2n} = \frac{1}{1+\Re\Phi_{2n+1}(0)}\Re\Big(\Phi_{2n+2}(0)(1-|\Phi_{2n+1}(0)|^2) - \overline{\Phi_{2n}(0)}(1+\Phi_{2n+1}(0))^2\Big), \quad n \ge 0, \tag{13}$$

$$2b_{2n+1} = \frac{1}{1+\Re\Phi_{2n+1}(0)}\Re\Big(\Phi_{2n}(0)(1-|\Phi_{2n+1}(0)|^2) - \overline{\Phi_{2n+2}(0)}(1+\Phi_{2n+1}(0))^2\Big), \quad n \ge 0, \tag{14}$$

$$2c_{2n+1} = \frac{\sqrt{1-|\Phi_{2n+1}(0)|^2}}{1+\Re\Phi_{2n+1}(0)}\Im\Big((1+\overline{\Phi_{2n+1}(0)})\Phi_{2n}(0) + \overline{\Phi_{2n+2}(0)}(1+\Phi_{2n+1}(0))\Big), \quad n \ge 0, \tag{15}$$

$$2c_{2n} = -\sqrt{\frac{1-|\Phi_{2n}(0)|^2}{(1+\Re\Phi_{2n-1}(0))(1+\Re\Phi_{2n+1}(0))}}\Im\Big((1+\Phi_{2n+1}(0))\overline{\Phi_{2n-1}(0)} + \Phi_{2n+1}(0)\Big), \quad n \ge 1. \tag{16}$$

Proof. For computing a_{n+2} take into account that

$$f_{2n}\left(\frac{\theta}{2}\right) = \frac{\sqrt{2}}{\sqrt{(1+\Re\Phi_{2n+1}(0))}\|\Phi_{2n}\|_\mu}\sin\left(n+\frac{1}{2}\right)\theta + ldt,$$

and

$$f_{2n-1}\left(\frac{\theta}{2}\right) = \frac{\sqrt{2(1+\Re\Phi_{2n-1}(0))}}{\|\Phi_{2n-1}\|_\mu}\cos\left(n-\frac{1}{2}\right)\theta + ldt.$$

In the even case, that is, if $n = 2k$ we identify in (10) the coefficients of $\sin(k+1+\frac{1}{2})\theta$ obtaining a_{2k+2} given by (11). In the odd case, that is, if $n = 2k-1$ we identify in (10) the coefficients of $\cos(k+\frac{1}{2})\theta$ obtaining a_{2k+1} given by (12).

For obtaining c_n we use that $c_n = \langle\cos\theta f_{n-1}(\frac{\theta}{2}), f_n(\theta)\rangle_\mu$. Taking into account relations (8) and (9) and distinguishing between even and odd terms we obtain:

$$c_{2n+1} = \left\langle\cos\theta f_{2n}\left(\frac{\theta}{2}\right), f_{2n+1}\left(\frac{\theta}{2}\right)\right\rangle_\mu = \frac{1}{2i\sqrt{(1+\Re(\Phi_{2n+1}(0)))^2}\|\Phi_{2n}\|_\mu\|\Phi_{2n+1}\|_\mu}$$
$$\times\left\langle\frac{(z+z^{-1})}{2}(z^{-(n-\frac{1}{2})}\Phi_{2n}(z) - z^{n-\frac{1}{2}}\overline{\Phi_{2n}}(z^{-1})), z^{-(n+\frac{1}{2})}\Phi_{2n+1}(z) + z^{n+\frac{1}{2}}\overline{\Phi_{2n+1}}(z^{-1})\right\rangle_\mu,$$

from which it follows (15), after using the orthogonality properties of the family $\{\Phi_n(z)\}$.

We proceed in the same way to obtain c_{2n}, that is,

$$c_{2n} = \left\langle \cos\theta f_{2n-1}\left(\frac{\theta}{2}\right), f_{2n}\left(\frac{\theta}{2}\right)\right\rangle_\mu = \frac{1}{2i\sqrt{(1+\Re(\Phi_{2n+1}(0)))(1+\Re(\Phi_{2n-1}(0)))}\|\Phi_{2n}\|_\mu\|\Phi_{2n-1}\|_\mu}$$
$$\times\left\langle \frac{(z+z^{-1})}{2}(z^{-(n-\frac{1}{2})}\Phi_{2n-1}(z)+z^{n-\frac{1}{2}}\overline{\Phi_{2n-1}}(z^{-1})), z^{-(n-\frac{1}{2})}\Phi_{2n}(z)-z^{n-\frac{1}{2}}\overline{\Phi_{2n}}(z^{-1})\right\rangle_\mu,$$

and applying the well-known properties of the sequence $\{\Phi_n(z)\}$ we obtain (16).

Finally, we get b_n in the following way

$$b_{2n-1} = \left\langle \cos\theta f_{2n-1}\left(\frac{\theta}{2}\right), f_{2n-1}\left(\frac{\theta}{2}\right)\right\rangle_\mu = \frac{1}{2(1+\Re(\Phi_{2n-1}(0)))\|\Phi_{2n-1}\|^2_\mu}$$
$$\times\left\langle \frac{(z+z^{-1})}{2}(z^{-(n-\frac{1}{2})}\Phi_{2n-1}(z)+z^{n-\frac{1}{2}}\overline{\Phi_{2n-1}}(z^{-1})), z^{-(n-\frac{1}{2})}\Phi_{2n-1}(z)+z^{n-\frac{1}{2}}\overline{\Phi_{2n-1}}(z^{-1})\right\rangle_\mu,$$

and

$$b_{2n} = \left\langle \cos\theta f_{2n}\left(\frac{\theta}{2}\right), f_{2n}\left(\frac{\theta}{2}\right)\right\rangle_\mu = \frac{1}{2(1+\Re(\Phi_{2n}(0)))\|\Phi_{2n}\|^2_\mu}$$
$$\times\left\langle \frac{(z+z^{-1})}{2}(z^{-(n-\frac{1}{2})}\Phi_{2n}(z)-z^{n-\frac{1}{2}}\overline{\Phi_{2n}}(z^{-1})), z^{-(n-\frac{1}{2})}\Phi_{2n}(z)-z^{n-\frac{1}{2}}\overline{\Phi_{2n}}(z^{-1})\right\rangle_\mu.$$

Using the orthogonality properties of the family $\{\Phi_n(z)\}$ we get (13) and (14). □

When the measure μ is symmetric we see in the next result that we recover the standard Szegő formula.

Corollary 1. *Let $\{f_n(\frac{\theta}{2})\}$ be the orthonormal sequence satisfying relation (10). Then*

(1) *It holds*

$$a_n > 0\ \forall n \geq 2, \quad b_n \in \mathbb{R}\ \forall n \geq 0, \quad c_n \in \mathbb{R}\ \forall n \geq 1.$$

(2) *If $\Phi_n(0) \in \mathbb{R}$, then we obtain a three term recurrence relation:*

(a) *If we denote the function $f_{2n}\left(\frac{\theta}{2}\right)$, in $\sin(k+\frac{1}{2})\theta$ for $k = 0,\ldots,n$, by $r_n(\frac{\theta}{2}) = f_{2n}(\frac{\theta}{2})$, then for $n \geq 1$*

$$\cos(\theta)r_n\left(\frac{\theta}{2}\right) = a_{2n+2}r_{n+1}\left(\frac{\theta}{2}\right) + b_{2n}r_n\left(\frac{\theta}{2}\right) + a_{2n}r_{n-1}\left(\frac{\theta}{2}\right). \tag{17}$$

(b) *If we denote the function $f_{2n-1}(\frac{\theta}{2})$, in $\cos(k+\frac{1}{2})\theta$ for $k = 0,\ldots,n-1$, by $q_n(\frac{\theta}{2}) = f_{2n-1}(\frac{\theta}{2})$, then we also obtain for $n \geq 2$*

$$\cos(\theta)q_n\left(\frac{\theta}{2}\right) = a_{2n+1}q_{n+1}\left(\frac{\theta}{2}\right) + b_{2n-1}q_n\left(\frac{\theta}{2}\right) + a_{2n-1}q_{n-1}\left(\frac{\theta}{2}\right). \tag{18}$$

Proof. 1. It is immediate from (11)–(16).

2. (a) Using (15) and (16) we obtain that the coefficients in relation (10) are $c_{2n+1} = c_{2n} = 0$. Moreover, using (11) and (13) we get that the other coefficients are given by

$$2a_{2n+2} = \sqrt{(1-\Phi^2_{2n+2}(0))(1-\Phi_{2n+1}(0))(1+\Phi_{2n+3}(0))}$$
$$2b_{2n} = \Phi_{2n+2}(0)(1-\Phi_{2n+1}(0)) - \Phi_{2n}(0)(1+\Phi_{2n+1}(0)).$$

(b) Proceeding in the same way we get that the coefficients in relation (10) are $c_{2n} = c_{2n-1} = 0$. Besides, the other coefficients are given by

$$2a_{2n+1} = \sqrt{(1-\Phi_{2n+1}(0))(1-\Phi^2_{2n}(0))(1+\Phi_{2n-1}(0))}$$
$$2b_{2n-1} = \Phi_{2n-2}(0)(1-\Phi_{2n-1}(0)) - \Phi_{2n}(0)(1+\Phi_{2n-1}(0)). \quad \square$$

Remark 3. From relation (17) we obtain the classical three term recurrence relation for OP on the interval, $p_n(x) = \frac{f_{2n}(\frac{\theta}{2})}{\sqrt{2}\sin\frac{\theta}{2}}$, and the coefficients $\{a_n\}$ and $\{b_n\}$ are related with the Verblunsky coefficients like in the transformation of measures given in [2,10].

From relation (18) can be obtained the three term recurrence relation satisfied by the OP on the interval, $p_{n-1}(x) = \frac{f_{2n-1}(\frac{\theta}{2})}{\sqrt{2}\cos\frac{\theta}{2}}$, and the coefficients $\{a_n\}$ and $\{b_n\}$ are related with the Verblunsky coefficients through the third transformation of measures, (see [2,10]).

Theorem 6. *Let $\{f_n(\frac{\theta}{2})\}$ be the sequence of trigonometric polynomials satisfying the five term recurrence relation (10) with sequences of coefficients $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$.*

If we know $f_0(\frac{\theta}{2}), f_1(\frac{\theta}{2})$ and $\|1\|_\mu$, and the above sequences $\{a_n\}_{n\geq 2}$, $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 1}$, then we can obtain $\{\Phi_n(0)\}$ for $n \geq 1$.

Proof. Indeed from $f_0(\frac{\theta}{2})$ we get $\Re(\Phi_1(0))$ and from $f_1(\frac{\theta}{2})$ we deduce $\Im(\Phi_1(0))$.

In order to obtain the two next Verblunsky parameters $\Phi_2(0)$ and $\Phi_3(0)$ we proceed as follows. Using the expressions of b_0 and c_1 given by (13) and (15) we obtain $\Re\Phi_2(0)$ and $\Im\Phi_2(0)$ respectively. Now, taking into account the expressions of a_2 and c_2 given by (11) and (16) we deduce $\Re\Phi_3(0)$ and $\Im\Phi_3(0)$ respectively.

If we proceed recursively in the same way, we obtain for every $n \geq 2$, from b_{2n-2} and c_{2n-1} given by (13) and (15), $\Re\Phi_{2n}(0)$ and $\Im\Phi_{2n}(0)$ respectively. Finally, using the expressions of a_{2n} and c_{2n} we get $\Re\Phi_{2n+1}(0)$ and $\Im\Phi_{2n+1}(0)$ respectively. □

Another important topic in the theory of OP is the study of the kernel polynomials. In the next theorem we obtain a Christoffel–Darboux type formula and the corresponding confluent form.

Theorem 7. *Let us denote by $K_N(\frac{\theta}{2},\frac{\tau}{2}) = \sum_{k=0}^{N} f_k(\frac{\theta}{2})f_k(\frac{\tau}{2})$ the reproducing kernel for the bi-orthogonal system. Then it holds that*

$$\begin{aligned}\frac{\cos\theta-\cos\tau}{2}K_N\left(\frac{\theta}{2},\frac{\tau}{2}\right) &= a_{N+2}\left(f_{N+2}\left(\frac{\theta}{2}\right)f_N\left(\frac{\tau}{2}\right)-f_{N+2}\left(\frac{\tau}{2}\right)f_N\left(\frac{\theta}{2}\right)\right)\\ &\quad + c_{N+1}\left(f_{N+1}\left(\frac{\theta}{2}\right)f_N\left(\frac{\tau}{2}\right)-f_{N+1}\left(\frac{\tau}{2}\right)f_N\left(\frac{\theta}{2}\right)\right)\\ &\quad + a_{N+1}\left(f_{N+1}\left(\frac{\theta}{2}\right)f_{N-1}\left(\frac{\tau}{2}\right)-f_{N+1}\left(\frac{\tau}{2}\right)f_{N-1}\left(\frac{\theta}{2}\right)\right),\end{aligned}\tag{19}$$

and we also have for equal arguments the confluent form given by:

$$\begin{aligned}\frac{\sin\theta}{2}K_N\left(\frac{\theta}{2},\frac{\theta}{2}\right) &= a_{N+2}\left(f'_{N+2}\left(\frac{\theta}{2}\right)f_N\left(\frac{\theta}{2}\right)-f_{N+2}\left(\frac{\theta}{2}\right)f'_N\left(\frac{\theta}{2}\right)\right)\\ &\quad + c_{N+1}\left(f'_{N+1}\left(\frac{\theta}{2}\right)f_N\left(\frac{\theta}{2}\right)-f_{N+1}\left(\frac{\theta}{2}\right)f'_N\left(\frac{\theta}{2}\right)\right)\\ &\quad + a_{N+1}\left(f'_{N+1}\left(\frac{\theta}{2}\right)f_{N-1}\left(\frac{\theta}{2}\right)-f_{N+1}\left(\frac{\theta}{2}\right)f'_{N-1}\left(\frac{\theta}{2}\right)\right).\end{aligned}\tag{20}$$

(The coefficients are given in (10).)

Proof. The steps of the proof are the same as in the standard case. First we write the recurrence relation (10) multiplied by $f_n(\frac{\tau}{2})$. Next we rewrite the same relation interchanging θ by τ and we subtract one from the other. Now we sum these relations from 0 up to N and simplify obtaining (19). □

Following the ideas given in [3,11] we connect in the next result the kernels related to the sequences $\{f_n(\frac{\theta}{2})\}$ and $\{\Phi_n(z)\}$.

Theorem 8. *Let $K_n(\frac{\theta}{2},\frac{\tau}{2})$ be the n-kernel function for the bi-orthogonal system $\{f_n(\frac{\theta}{2})\}$ and let $S_n(z,y)$ be the n-kernel function for the orthogonal system $\{\Phi_n(z)\}$ defined as usual, that is, $S_n(z,y) = \sum_{k=0}^{n}\frac{\Phi_k(z)\overline{\Phi_k(y)}}{\|\Phi_k(z)\|^2}$. If $z = \mathrm{e}^{\mathrm{i}\theta}$ and $y = \mathrm{e}^{\mathrm{i}\tau}$ then it holds that*

$$K_{2n-1}\left(\frac{\theta}{2},\frac{\tau}{2}\right) = (y\bar{z})^{n-\frac{1}{2}}S_{2n-1}(z,y).$$

Proof. Take into account that $(y\bar{z})^{n-\frac{1}{2}}S_{2n-1}(z,y)$ has the reproducing property on $z^{k+\frac{1}{2}} \pm z^{-(k+\frac{1}{2})}$ for $k = 0,\ldots,n-1$, that is,

$$\left\langle (y\bar{z})^{n-\frac{1}{2}}S_{2n-1}(z,y), z^{k+\frac{1}{2}} \pm z^{-(k+\frac{1}{2})}\right\rangle_\mu = (\bar{y})^{k+\frac{1}{2}} \pm (\bar{y})^{-(k+\frac{1}{2})}, \quad k = 0,\ldots,n-1. \quad \square$$

Using a different approach, similar results to those given in Theorems 7 and 8 are also obtained in [6,8].

4. A Favard's type theorem

Next we are going to present another new important recurrence relation satisfied by the sequence $\{f_n(\frac{\theta}{2})\}$. It will be used together with (10) to obtain the main result of this section, the Favard's type theorem.

Theorem 9. *Let $\{f_n(\frac{\theta}{2})\}$ be the sequence of trigonometric orthonormal functions. Then there exist four sequences of coefficients $\{\alpha_{2n+1}\}_{n\geq 1}$, $\{\tilde{a}_n\}_{n\geq 2}$, $\{\tilde{b}_n\}_{n\geq 0}$, $\{\tilde{c}_n\}_{n\geq 1}$ such that for $n \geq 0$ the following seven term recurrence relation holds:*

$$\begin{aligned}\sin(\theta)f_n\left(\frac{\theta}{2}\right) &= \alpha_{n+3}f_{n+3}\left(\frac{\theta}{2}\right)+\tilde{a}_{n+2}f_{n+2}\left(\frac{\theta}{2}\right)+\tilde{c}_{n+1}f_{n+1}\left(\frac{\theta}{2}\right)+\tilde{b}_nf_n\left(\frac{\theta}{2}\right)\\ &\quad +\tilde{c}_nf_{n-1}\left(\frac{\theta}{2}\right)+\tilde{a}_nf_{n-2}\left(\frac{\theta}{2}\right)+\alpha_nf_{n-3}\left(\frac{\theta}{2}\right).\end{aligned}\tag{21}$$

Since $\alpha_{n+3} = 0$ if n is odd and $\alpha_n = 0$ if n is even, indeed we have a six term recurrence relation with the initial conditions $f_{-2}(\frac{\theta}{2}) = 0$, $f_{-1}(\frac{\theta}{2}) = 0$, $f_0(\frac{\theta}{2}) = \frac{1}{\sqrt{n_0}} \sin\frac{\theta}{2}$, *and* $f_1(\frac{\theta}{2}) = \sqrt{\frac{n_0}{n_0 n_2 - n_1^2}}(\cos\frac{\theta}{2} - \frac{n_1}{n_0}\sin\frac{\theta}{2})$, *where* $n_0 = \langle \sin\frac{\theta}{2}, \sin\frac{\theta}{2}\rangle_\mu$, $n_1 = \langle \sin\frac{\theta}{2}, \cos\frac{\theta}{2}\rangle_\mu$ *and* $n_2 = \langle \cos\frac{\theta}{2}, \cos\frac{\theta}{2}\rangle_\mu$.

Proof. In order to determine the number of terms we distinguish between the even and odd cases. We write $\sin(\theta) f_{2n}(\frac{\theta}{2}) = \sum_{k=0}^{2n+3} a_{k,2n} f_k(\frac{\theta}{2})$ and $\sin(\theta) f_{2i}(\frac{\theta}{2}) = \sum_{k=0}^{2i+3} a_{k,2i} f_k(\frac{\theta}{2})$, from which it follows

$$a_{2i,2n} = \left\langle \sin(\theta) f_{2n}\left(\frac{\theta}{2}\right), f_{2i}\left(\frac{\theta}{2}\right)\right\rangle_\mu = \left\langle \sin(\theta) f_{2i}\left(\frac{\theta}{2}\right), f_{2n}\left(\frac{\theta}{2}\right)\right\rangle_\mu = a_{2n,2i}.$$

Since $a_{2n,2i} = 0$ if $2n > 2i+3 \Leftrightarrow 2i < 2n-3$, then $a_{2n-4,2n} = \cdots = a_{0,2n} = 0$.

Now, using that $\sin(\theta) f_{2i+1}(\frac{\theta}{2}) = \sum_{k=0}^{2i+3} a_{k,2i+1} f_k(\frac{\theta}{2})$ and proceeding in the same way

$$a_{2i+1,2n} = \left\langle \sin(\theta) f_{2n}\left(\frac{\theta}{2}\right), f_{2i+1}\left(\frac{\theta}{2}\right)\right\rangle_\mu = \left\langle \sin(\theta) f_{2i+1}\left(\frac{\theta}{2}\right), f_{2n}\left(\frac{\theta}{2}\right)\right\rangle_\mu = a_{2n,2i+1}.$$

Since $a_{2n,2i+1} = 0$ if $2n > 2i+3 \Leftrightarrow 2i+1 < 2n-2$, then $a_{2n-3,2n} = \cdots = a_{1,2n} = 0$.

Hence

$$\sin(\theta) f_{2n}\left(\frac{\theta}{2}\right) = \sum_{k=2n-2}^{2n+3} a_{k,2n} f_k\left(\frac{\theta}{2}\right).$$

Reasoning in the same way in the odd case we get

$$\sin(\theta) f_{2n+1}\left(\frac{\theta}{2}\right) = \sum_{k=2n-2}^{2n+3} a_{k,2n+1} f_k\left(\frac{\theta}{2}\right).$$

Since $a_{n+3,n} = a_{n,n+3}$, $a_{n+2,n} = a_{n,n+2}$ and $a_{n+1,n} = a_{n,n+1}$, we simplify the notation writing

$$a_{n+3,n} = \alpha_{n+3},\ a_{n+2,n} = \tilde{a}_{n+2},\ a_{n+1,n} = \tilde{c}_{n+1},\ a_{n,n} = \tilde{b}_n.$$

Therefore we obtain

$$\begin{aligned}
\sin(\theta) f_{2n}\left(\frac{\theta}{2}\right) &= \alpha_{2n+3} f_{2n+3}\left(\frac{\theta}{2}\right) + \tilde{a}_{2n+2} f_{2n+2}\left(\frac{\theta}{2}\right) + \tilde{c}_{2n+1} f_{2n+1}\left(\frac{\theta}{2}\right) \\
&\quad + \tilde{b}_{2n} f_{2n}\left(\frac{\theta}{2}\right) + \tilde{c}_{2n} f_{2n-1}\left(\frac{\theta}{2}\right) + \tilde{a}_{2n} f_{2n-2}\left(\frac{\theta}{2}\right), \\
\sin(\theta) f_{2n+1}\left(\frac{\theta}{2}\right) &= \tilde{a}_{2n+3} f_{2n+3}\left(\frac{\theta}{2}\right) + \tilde{c}_{2n+2} f_{2n+2}\left(\frac{\theta}{2}\right) + \tilde{b}_{2n+1} f_{2n+1}\left(\frac{\theta}{2}\right) \\
&\quad + \tilde{c}_{2n+1} f_{2n}\left(\frac{\theta}{2}\right) + \tilde{a}_{2n+1} f_{2n-1}\left(\frac{\theta}{2}\right) + \alpha_{2n+1} f_{2n-2}\left(\frac{\theta}{2}\right).
\end{aligned}$$

Besides, we have

$$\begin{aligned}
&|\alpha_{2n+3}|^2 + |\tilde{a}_{2n+2}|^2 + |\tilde{c}_{2n+1}|^2 + |\tilde{b}_{2n}|^2 + |\tilde{c}_{2n}|^2 + |\tilde{a}_{2n}|^2 < 1, \\
&|\tilde{a}_{2n+3}|^2 + |\tilde{c}_{2n+2}|^2 + |\tilde{b}_{2n+1}|^2 + |\tilde{c}_{2n+1}|^2 + |\tilde{a}_{2n+1}|^2 + |\alpha_{2n+1}|^2 < 1. \quad \square
\end{aligned}$$

Corollary 2. *Let $\{f_n(\frac{\theta}{2})\}$ be the sequence of trigonometric orthonormal functions. Then the coefficients of relations (10) and (21) satisfy that*

$$|a_{2n+2}|^2 + |c_{2n+1}|^2 + |b_{2n}|^2 + |c_{2n}|^2 + |a_{2n}|^2 + |\alpha_{2n+3}|^2 + |\tilde{a}_{2n+2}|^2 + |\tilde{c}_{2n+1}|^2 + |\tilde{b}_{2n}|^2 + |\tilde{c}_{2n}|^2 + |\tilde{a}_{2n}|^2 = 1,$$

and

$$\begin{aligned}
&|a_{2n+3}|^2 + |c_{2n+2}|^2 + |b_{2n+1}|^2 + |c_{2n+1}|^2 + |a_{2n+1}|^2 + |\tilde{a}_{2n+3}|^2 + |\tilde{c}_{2n+2}|^2 + |\tilde{b}_{2n+1}|^2 \\
&\quad + |\tilde{c}_{2n+1}|^2 + |\tilde{a}_{2n+1}|^2 + |\alpha_{2n+1}|^2 = 1.
\end{aligned}$$

Proof. It is immediate taking into account that the sum of the squares of the coefficients is given by

$$\begin{aligned}
\left\langle \cos\theta f_n\left(\frac{\theta}{2}\right), \cos\theta f_n\left(\frac{\theta}{2}\right)\right\rangle_\mu + \left\langle \sin\theta f_n\left(\frac{\theta}{2}\right), \sin\theta f_n\left(\frac{\theta}{2}\right)\right\rangle_\mu &= \int_{-\pi}^{\pi} \left(\cos^2\theta f_n^2\left(\frac{\theta}{2}\right) + \sin^2\theta f_n^2\left(\frac{\theta}{2}\right)\right) \mathrm{d}\mu \\
&= \int_{-\pi}^{\pi} f_n^2\left(\frac{\theta}{2}\right) \mathrm{d}\mu = 1. \quad \square
\end{aligned}$$

Like in the recurrence relation (10) we can relate the Verblunsky coefficients and the parameters of the six term recurrence relation. Next we give the corresponding formulas distinguishing between even and odd terms.

Theorem 10. *Let $\{f_n(\frac{\theta}{2})\}$ be the orthonormal sequence satisfying relation (21) and let $\{\Phi_n(z)\}$ be the MOPS(μ). Then the following relations hold*

$$2\alpha_{2n+3} = -\sqrt{\frac{(1-|\Phi_{2n+3}(0)|^2)(1-|\Phi_{2n+2}(0)|^2)(1-|\Phi_{2n+1}(0)|^2)}{(1+\Re\Phi_{2n+1}(0))(1+\Re\Phi_{2n+3}(0))}}, \quad n \geq 0, \tag{22}$$

$$2\tilde{a}_{2n+3} = -\sqrt{\frac{(1-|\Phi_{2n+3}(0)|^2)(1-|\Phi_{2n+2}(0)|^2)}{(1+\Re\Phi_{2n+1}(0))(1+\Re\Phi_{2n+3}(0))}}\Im\Phi_{2n+1}(0), \quad n \geq 0, \tag{23}$$

$$2\tilde{a}_{2n+2} = \sqrt{\frac{(1-|\Phi_{2n+2}(0)|^2)(1-|\Phi_{2n+1}(0)|^2)}{(1+\Re\Phi_{2n+1}(0))(1+\Re\Phi_{2n+3}(0))}}\Im\Phi_{2n+3}(0), \quad n \geq 0, \tag{24}$$

$$2\tilde{b}_{2n+1} = \frac{1}{1+\Re\Phi_{2n+1}(0)}\Im\left(\overline{\Phi_{2n}(0)}(1-|\Phi_{2n+1}(0)|^2) + \overline{\Phi_{2n+2}(0)}(1+\Phi_{2n+1}(0))^2\right), \quad n \geq 0, \tag{25}$$

$$2\tilde{b}_{2n} = \frac{1}{1+\Re\Phi_{2n+1}(0)}\Im\left(\Phi_{2n+2}(0)(1-|\Phi_{2n+1}(0)|^2) + \Phi_{2n}(0)(1+\overline{\Phi_{2n+1}(0)})^2\right), \quad n \geq 0, \tag{26}$$

$$2\tilde{c}_{2n+1} = \frac{\sqrt{1-|\Phi_{2n+1}(0)|^2}}{1+\Re\Phi_{2n+1}(0)}\Re\Big((1+\overline{\Phi_{2n+1}(0)})(\Phi_{2n}(0)+\Phi_{2n+2}(0))\Big), \quad n \geq 0, \tag{27}$$

$$2\tilde{c}_{2n+2} = \sqrt{\frac{1-|\Phi_{2n+2}(0)|^2}{(1+\Re\Phi_{2n+1}(0))(1+\Re\Phi_{2n+3}(0))}}\Re\Big((1+\Phi_{2n+3}(0))(1+\overline{\Phi_{2n+1}(0)})\Big), \quad n \geq 0. \tag{28}$$

Proof. We proceed in the same way as in Theorem 5, using the recurrence relation (21) and the orthogonality properties of the sequence $\{\Phi_n(z)\}$. □

Finally, we present a Favard's type theorem. In the sequel we are going to denote by T the space of trigonometric polynomials and by Z the space generated by the functions of the second kind trigonometric system, that is, the space of the second kind trigonometric functions.

Lemma 3. *Let $\{f_n\left(\frac{\theta}{2}\right)\} \subset Z$ be a sequence of second kind trigonometric functions satisfying the recurrence relations (10) and (21) with their corresponding initial conditions:*

$$\cos\theta f_n\left(\frac{\theta}{2}\right) = a_{n+2}f_{n+2}\left(\frac{\theta}{2}\right) + c_{n+1}f_{n+1}\left(\frac{\theta}{2}\right) + b_nf_n\left(\frac{\theta}{2}\right) + c_nf_{n-1}\left(\frac{\theta}{2}\right) + a_nf_{n-2}\left(\frac{\theta}{2}\right),$$

$$\begin{aligned}\sin(\theta)f_n\left(\frac{\theta}{2}\right) &= \alpha_{n+3}f_{n+3}\left(\frac{\theta}{2}\right) + \tilde{a}_{n+2}f_{n+2}\left(\frac{\theta}{2}\right) + \tilde{c}_{n+1}f_{n+1}\left(\frac{\theta}{2}\right)\\ &\quad + \tilde{b}_nf_n\left(\frac{\theta}{2}\right) + \tilde{c}_nf_{n-1}\left(\frac{\theta}{2}\right) + \tilde{a}_nf_{n-2}\left(\frac{\theta}{2}\right) + \alpha_nf_{n-3}\left(\frac{\theta}{2}\right),\end{aligned}$$

with $\alpha_{n+3} = 0$ if n is odd and $\alpha_n = 0$ if n is even, and $f_{-2} = 0, f_{-1} = 0, f_0 = \eta_0\sin\frac{\theta}{2}$, and $f_1 = \eta_1\cos\frac{\theta}{2} + a\sin\frac{\theta}{2}$. Then there exists an inner product $\langle,\rangle$ defined on $Z \times Z$ such that

$$\left\langle f_n\left(\frac{\theta}{2}\right), f_m\left(\frac{\theta}{2}\right)\right\rangle = \delta_{n,m}, \quad \forall n, m \tag{29}$$

and

$$\left\langle P(\theta)f_n\left(\frac{\theta}{2}\right), f_m\left(\frac{\theta}{2}\right)\right\rangle = \left\langle f_n\left(\frac{\theta}{2}\right), P(\theta)f_m\left(\frac{\theta}{2}\right)\right\rangle, \quad \forall n, m, \quad \text{and} \quad \forall P(\theta) \in T. \tag{30}$$

Proof. It is possible to construct an inner product on $Z \times Z$ defined on the basis of Z by

$$\left\langle f_i\left(\frac{\theta}{2}\right), f_j\left(\frac{\theta}{2}\right)\right\rangle = \delta_{i,j} \quad \text{for all } i, j.$$

Moreover, using the recurrence relations it is easy to obtain that the inner product satisfies property (30). □

Theorem 11. *Let $\{f_n\left(\frac{\theta}{2}\right)\} \subset Z$ be a sequence of second kind trigonometric functions satisfying the recurrence relations (10) and (21) with their corresponding initial conditions. Then there exists a unique positive measure μ on $[-\pi,\pi]$ such that $\int_{-\pi}^{\pi} f_n\left(\frac{\theta}{2}\right)f_m\left(\frac{\theta}{2}\right)\mathrm{d}\mu(\theta) = \delta_{n,m}$.*

Proof. Let us consider the space of trigonometric polynomials T, which can be considered as the space generated as follows $T =< \{1, \sin\frac{\theta}{2}\cos(k+\frac{1}{2})\theta, \cos\frac{\theta}{2}\cos(k+\frac{1}{2})\theta : k = 0, 1, \ldots\} >$. We define the following functional $\mathcal{L}$ on T using the inner product defined in Lemma 3.

$$\mathcal{L}(P(\theta)) := \left\langle \cos\frac{\theta}{2}P(\theta), \cos\frac{\theta}{2}\right\rangle + \left\langle \sin\frac{\theta}{2}P(\theta), \sin\frac{\theta}{2}\right\rangle.$$

By the properties of the inner product, we have that $\mathcal{L}$ is linear. Next we are going to prove that $\mathcal{L}$ is a bounded positive definite functional. We begin with the positive definite character. Now let us assume that $P(\theta) \in T$ is a positive trigonometric polynomial. Taking into account that $P(\theta)$ can be written like $P(\theta) = (Q_1(\theta))^2 + (Q_2(\theta))^2$, where Q_1, Q_2 are trigonometric polynomials with real coefficients, it is easy to see that $\mathcal{L}(P(\theta)) > 0$. Hence it suffices to prove that $\mathcal{L}((Q(\theta))^2) > 0$ for $Q(\theta)$ a trigonometric polynomial with real coefficients, $Q \neq 0$. Indeed applying the definition of $\mathcal{L}$ and the property of the inner product given in (30), we have

$$\begin{aligned}\mathcal{L}((Q(\theta))^2) &= \left\langle \cos\frac{\theta}{2}(Q(\theta))^2, \cos\frac{\theta}{2}\right\rangle + \left\langle \sin\frac{\theta}{2}(Q(\theta))^2, \sin\frac{\theta}{2}\right\rangle \\ &= \left\langle \cos\frac{\theta}{2}Q(\theta), \cos\frac{\theta}{2}Q(\theta)\right\rangle + \left\langle \sin\frac{\theta}{2}Q(\theta), \sin\frac{\theta}{2}Q(\theta)\right\rangle > 0.\end{aligned}$$

Therefore $\mathcal{L}$ is definite positive on T. Now it is immediate to obtain that $\mathcal{L}$ is bounded. Indeed let P be a trigonometric polynomial. Then $\|P\|_\infty \pm P(\theta) \geq 0 \;\; \forall \theta \in [-\pi, \pi]$ and since $\mathcal{L}$ is definite positive then $|\mathcal{L}(P)| \leq \mathcal{L}(1)\|P\|_\infty$, which proves the continuity of $\mathcal{L}$. If we apply the Hahn–Banach theorem we get that there exists a unique bounded and positive linear extension $\tilde{\mathcal{L}} : C[-\pi, \pi] \to \mathbb{R}$ with the same norm and such that $\tilde{\mathcal{L}}(P) = \mathcal{L}(P), \;\; \forall P \in T$. Finally, applying the Riesz representation theorem for linear and positive functionals we get that there exists a unique positive measure μ such that $\tilde{\mathcal{L}}(f) = \int_{-\pi}^{\pi} f(\theta)\mathrm{d}\mu(\theta)$ for $f \in C[-\pi, \pi]$. In particular $\mathcal{L}(P(\theta)) = \tilde{\mathcal{L}}(P(\theta)) = \int_{-\pi}^{\pi} P(\theta)\mathrm{d}\mu(\theta) \;\forall P \in T$. Now let us see that the inner product $\langle, \rangle$ can also be represented. Let $f\left(\frac{\theta}{2}\right), g\left(\frac{\theta}{2}\right) \in Z$, then $f\left(\frac{\theta}{2}\right) g\left(\frac{\theta}{2}\right) \in T$ and

$$\begin{aligned}\mathcal{L}\left(f\left(\frac{\theta}{2}\right)g\left(\frac{\theta}{2}\right)\right) &= \left\langle \cos\frac{\theta}{2}f\left(\frac{\theta}{2}\right)g\left(\frac{\theta}{2}\right), \cos\frac{\theta}{2}\right\rangle + \left\langle \sin\frac{\theta}{2}f\left(\frac{\theta}{2}\right)g\left(\frac{\theta}{2}\right), \sin\frac{\theta}{2}\right\rangle \\ &= \left\langle f\left(\frac{\theta}{2}\right), \cos^2\frac{\theta}{2}g\left(\frac{\theta}{2}\right)\right\rangle + \left\langle f\left(\frac{\theta}{2}\right), \sin^2\frac{\theta}{2}g\left(\frac{\theta}{2}\right)\right\rangle = \left\langle f\left(\frac{\theta}{2}\right), g\left(\frac{\theta}{2}\right)\right\rangle.\end{aligned}$$

On the other hand, $\mathcal{L}\left(f\left(\frac{\theta}{2}\right)g\left(\frac{\theta}{2}\right)\right) = \int_{-\pi}^{\pi} f\left(\frac{\theta}{2}\right)g\left(\frac{\theta}{2}\right)d\mu(\theta)$ and therefore

$$\left\langle f\left(\frac{\theta}{2}\right), g\left(\frac{\theta}{2}\right)\right\rangle = \int_{-\pi}^{\pi} f\left(\frac{\theta}{2}\right)g\left(\frac{\theta}{2}\right)\mathrm{d}\mu(\theta).$$

In particular we have the result for the sequence $\{f_n\left(\frac{\theta}{2}\right)\}$, that is,

$$\left\langle f_n\left(\frac{\theta}{2}\right), f_m\left(\frac{\theta}{2}\right)\right\rangle = \int_{-\pi}^{\pi} f_n\left(\frac{\theta}{2}\right)f_m\left(\frac{\theta}{2}\right)\mathrm{d}\mu(\theta). \quad \square$$

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References

[1] G. Szegő, Orthogonal Polynomials, fourth ed., in: Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, RI, 1975.
[2] J. García-Amor, Ortogonalidad Bernstein-Chebyshev en la recta real, Doctoral Dissertation, Universidad de Vigo, 2003 (in Spanish).
[3] G. Szegő, On bi-orthogonal systems of trigonometric polynomials, Magyar Tud. Alcad. Kutató Int. Kőzl. 8 (1963) 255–273.
[4] P.J. Davis, Interpolation and Approximation, Dover Publications, New York, 1975.
[5] A. Zygmund, Trigonometric series, second ed., Cambridge University Press, 1988.
[6] R. Cruz-Barroso, P. González-Vera, O. Njastad, On bi-orthogonal systems of trigonometric functions and quadrature formulas for periodic integrands, Numer. Algorithms 44 (2007) 309–333.
[7] R. Cruz-Barroso, L. Daruis, P. González-Vera, O. Njastad, Sequences of orthogonal Laurent polynomials, bi-orthogonality, and quadrature formulas on the unit circle, J. Comput. Appl. Math. 200 (2007) 424–440.
[8] R. Cruz-Barroso, Sobre polinomios de Laurent ortogonales y fórmulas de cuadratura en la circunferencia unidad, Doctoral Dissertation, Universidad de La Laguna, 2007 (in Spanish).
[9] B. Simon, Orthogonal Polynomials on the Unit Circle, in: Amer. Math. Soc. Colloq. Publ., vol. 54, Amer. Math. Soc, Providence, RI, 2005.
[10] E. Berriochoa, A. Cachafeiro, J. García-Amor, Connection between orthogonal polynomials on the unit circle and bounded interval, J. Comput. Appl. Math. 177 (1) (2005) 205–223.
[11] E. Berriochoa, A. Cachafeiro, J. García-Amor, A system of biorthogonal trigonometric polynomials, in: Difference Equations, Special Functions, and Orthogonal Polynomials, in: S. Elaydi, et al. (Eds), Proceedings of the International Conference Munich 2005, World Scientific, Singapore, 2007, pp. 80–89.